Semester One, 2011
Question/Answer Booklet

MATHEMATICS SPECIALIST 3CD

Please place your student identification label in this box

Section Two: Calculator-assumed

SOLUTIONS

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time for this section:	eighty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on **one** unfolded sheet of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available
Section One: Calculator-free	6	6	40	35
Section Two: Calculator-assumed	11	11	80	65
			Total	100

Instructions to candidates

1. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

3. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(65 Marks)**

This section has **11** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Working time: 80 minutes.

Question 7 **(8 marks)**

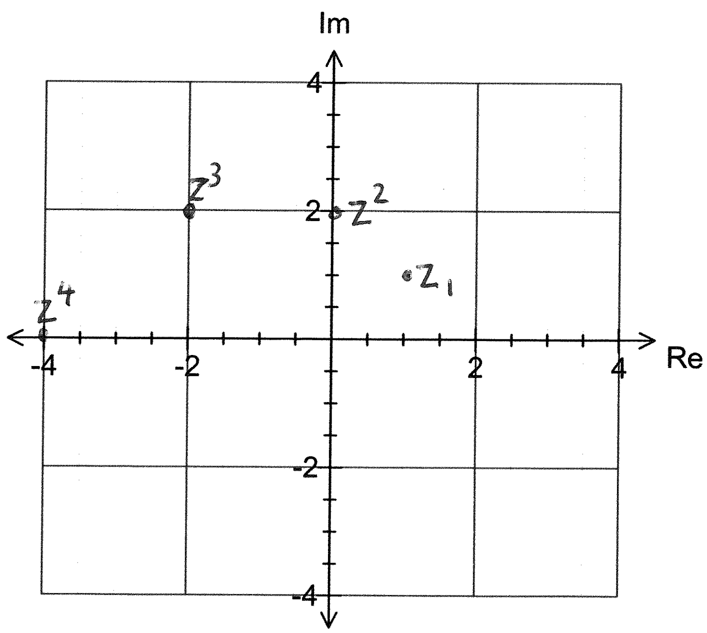
If $z = 1 + i$, simplify

(a) $z^2 = 2i$ (1 mark)

(b) $z^3 = -2 + 2i$ (1 mark)

(c) $z^4 = -4$ (1 mark)

(d) Plot z, z^2, z^3 and z^4 on the Argand diagram below. (2 marks)



(e) Multiplying z by $(1 + i)$ has the effect of rotating z about the origin. Describe this rotation by stating the angle of rotation and direction of rotation. (1 mark)

$45°$ anticlockwise ✓

(f) Hence, describe the transformation effect when a complex number $z = \sqrt{3} + i$ is multiplied by itself each time. (2 marks)

$\arg z = \frac{\pi}{6} = 30°$ ✓

$\therefore$ rotation $30°$ anticlockwise ✓

Question 8 **(4 marks)**

Prove: $\dfrac{cosx}{1 - tanx} + \dfrac{sin^2 x}{sinx - cosx} = sinx + cosx$

$$L.S = \frac{\cos x}{1 - \frac{\sin x}{\cos x}} + \frac{\sin^2 x}{\sin x - \cos x} \quad ✓$$

$$= \frac{\cos x}{\frac{\cos x - \sin x}{\cos x}} + \frac{\sin^2 x}{\sin x - \cos x}$$

$$= \frac{\cos^2 x}{\cos x - \sin x} - \frac{\sin^2 x}{\cos x - \sin x} \quad ✓$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}$$

$$= \frac{\cancel{(\cos x - \sin x)}(\cos x + \sin x)}{\cancel{\cos x - \sin x}} \quad ✓$$

$$= \sin x + \cos x \quad ✓$$

$$= R.S.$$

Question 9 **(7 marks)**

(a) The position vectors of points B and C are $\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{i} - \mathbf{j}$ respectively relative to an origin O. Point N lies on BC such that BN:NC = 2:3. Find the position vector of N.

(2 marks)

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$

$= \langle 1,-1,5 \rangle + \frac{2}{5}\overrightarrow{BC}$

$= \langle 1,-1,5 \rangle + \frac{2}{5}(\langle 1,-1,0 \rangle - \langle 1,-1,5 \rangle)$ ✓

$= \langle 1,-1,7 \rangle$ ✓

(b) Find the vector equation of the plane in the form $\mathbf{r} \bullet \mathbf{n} = \mathbf{c}$ passing through (3, -1, 0) and containing the line $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ with $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ as its normal.

(2 marks)

$\underline{r} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ ✓

$\underline{r} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = 13$ ✓

(b) The plane π has equation $\mathbf{r} \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 15$. Show that the line $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ is parallel to the plane π but does not lie in π.

(3 marks)

$\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 3 + 2 - 5$

$= 0$ ✓

$\therefore \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \perp \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ ✓

∴ line is parallel to plane

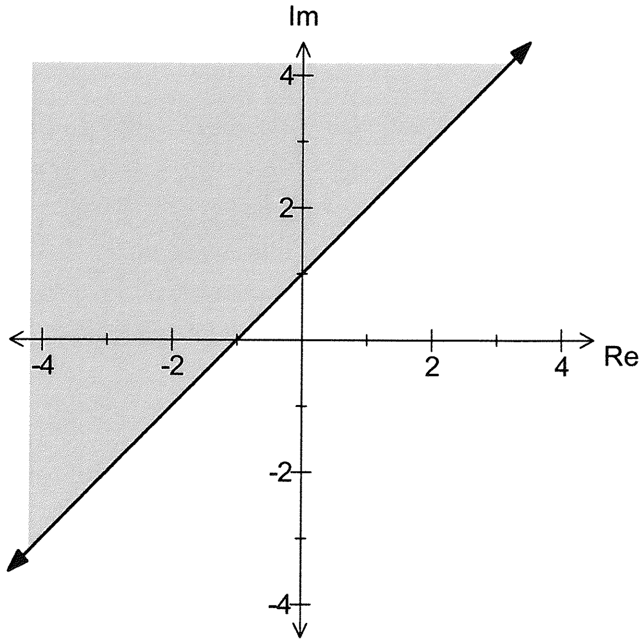
$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ lies on the line

but $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 9 \neq 15$ ✓

$\therefore \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ does not lie on the plane

Question 10 **(5 marks)**

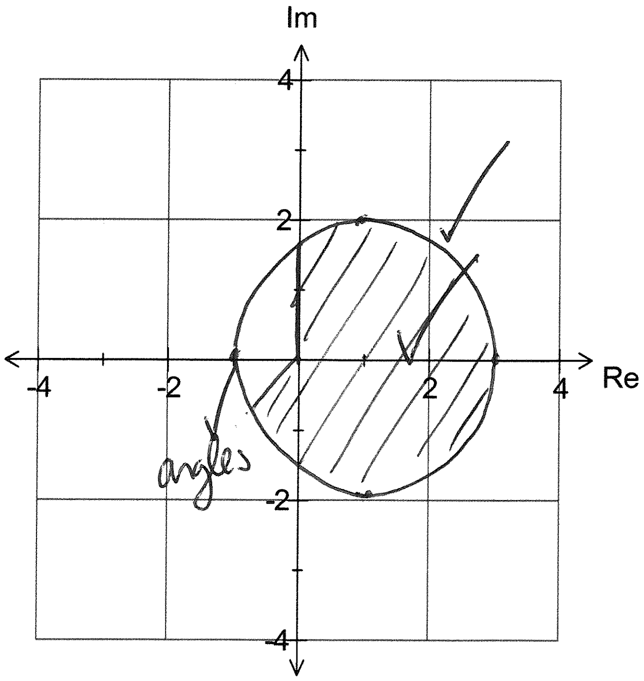
(a) State the inequality that describes the set of points in the complex plane, illustrated by the shading below. (2 marks)



$|z| \geq |z-(-1+i)|$

(b) Sketch on the complex plane below, the region defined by:

$-\frac{3\pi}{4} \leq arg(z) \leq \frac{\pi}{2}$ and $|z-1| \leq 2$ (3 marks)



Question 11 **(6 marks)**

(a) Find the equation of the line L passing through the points A and B with position vectors $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ respectively. (2 marks)

$$\underline{d} = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \checkmark$$

$$\therefore \underline{r} = \begin{pmatrix}2\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix} \checkmark$$

(b) Find the point of intersection, D, of line L from part (a) and the plane P such that the vector equation of P is $\mathbf{r} \bullet \begin{pmatrix}2\\3\\-1\end{pmatrix} = -2$. (2 marks)

$$\left[\begin{pmatrix}2\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}2\\3\\-1\end{pmatrix} = -2 \checkmark$$

$$\lambda = -1$$

$$\therefore \text{pt } (1, -1, 1) \checkmark$$

(c) Find the angle (to the nearest degree) between the line L and the plane at D. (2 marks)

angle between $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\1\end{pmatrix} = 40° \checkmark$

$\therefore$ angle between line & plane $= 90 - 40$
$= 50° \checkmark$

Question 12 **(4 marks)**

The gradient of a curve at any point is given by $\frac{dy}{dx} = 2 - \frac{x^3}{8}$. The curve intersects the x-axis at the point P. Given that the gradient of the curve at P is 1, find the equation of the curve.

$$\int 2 - \frac{x^3}{8}\,dx = 2x - \frac{x^4}{32} + c \quad ✓$$

at $P(x,0)$ $\frac{dy}{dx} = 1$

$$\therefore \quad 2 - \frac{x^3}{8} = 1$$

$$x = 2 \quad ✓$$

$\therefore$ at $(2,0)$ $\quad 2(2) - \frac{2^4}{32} + c = 0$

$$c = \frac{7}{2} \quad ✓$$

$\therefore$ curve $y = 2x - \frac{x^4}{32} + \frac{7}{2}$ ✓

Question 13 **(5 marks)**

(a) Show that $\frac{dy}{dx} = \frac{4}{y-1}$ for $y^2 = 2y + 8x - 17$. (2 marks)

$$2y\frac{dy}{dx} = 2\frac{dy}{dx} + 8$$

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} = 8$$

$$\frac{dy}{dx}(y-1) = 4$$

$$dy/dx = \frac{4}{y-1}$$

(b) Show that the tangents to the curve $y^2 = 2y + 8x - 17$ at the points where $x = 4$ are perpendicular. (3 marks)

$$x = 4 \Rightarrow y^2 = 2y + 32 - 17$$

$$y^2 - 2y - 15 = 0$$

$$(y-5)(y+3) = 0$$

$$y = 5, -3$$

$$\left.\frac{dy}{dx}\right|_{y=5} = \frac{4}{5-1} = 1$$

$$\left.\frac{dy}{dx}\right|_{y=3} = \frac{4}{-3-1} = -1$$

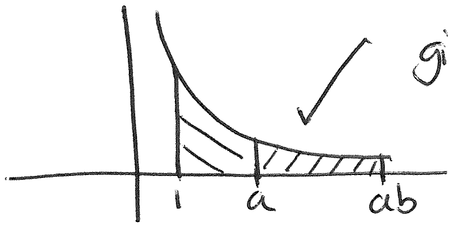
$$\therefore m_1 \times m_2 = 1 \times -1$$
$$= -1$$

$\therefore$ tangents perpendicular

Question 14 **(5 marks)**

The natural logarithm can be expressed as $\ln x = \int_1^x \frac{1}{t}\,dt$

(a) Explain graphically why $\int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$, where $1 < a < b$ and a and b are constants. (2 marks)



✓ given $1 < a < b \Rightarrow a < ab$ ✓

total shaded area $= \int_1^{ab} \frac{1}{t}\,dt$

(b) Use the integral definition of the natural logarithm, and where required, the substitution $u = \frac{bt}{a}$, to deduce $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$. (3 marks)

$$\ln\left(\frac{a}{b}\right) = \int_1^{\frac{a}{b}} \frac{1}{t}\,dt$$

$$= \int_1^{a} \frac{1}{t}\,dt + \int_a^{\frac{a}{b}} \frac{1}{t}\,dt \checkmark$$

$u = \frac{bt}{a}$ ✓

$\frac{du}{dt} = \frac{b}{a}$

$t = \frac{a}{b}$ $u = 1$

$t = a$ $u = b$

$$= \ln a + \int_b^{1} \frac{1}{\frac{au}{b}} \cdot \frac{a}{b}\,du$$

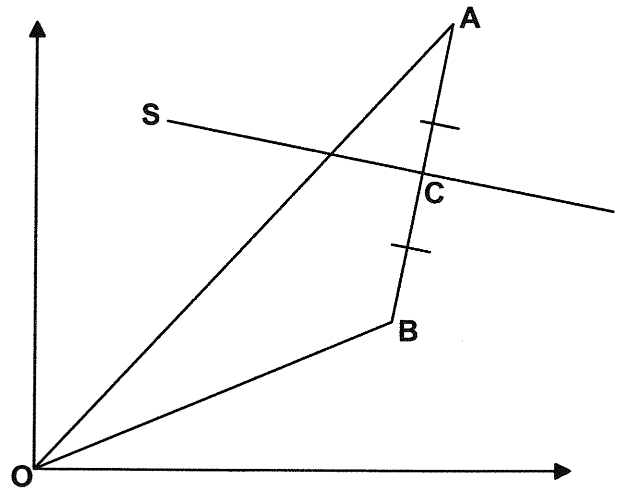
$$= \ln a - \int_1^{b} \frac{\cancel{a}}{\cancel{b}} \cdot \frac{\cancel{b}}{au}\,du$$

$$= \ln a - \int_1^{b} \frac{1}{u}\,du \checkmark$$

$$= \ln a - \ln b$$

Question 15 **(6 marks)**

O, A and B are non-collinear points on a plane. S is an arbitrary point on the perpendicular bisector of $\overrightarrow{AB}$ and C is the point of intersection of this bisector and $\overrightarrow{AB}$.
Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OS} = \mathbf{s}$, $\overrightarrow{OC} = \mathbf{c}$.



(a) Write $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ (1 mark)

$$\overrightarrow{OC} = \underline{b} + \frac{1}{2}\overrightarrow{BA}$$
$$= \frac{1}{2}(\underline{a} + \underline{b}) \checkmark$$

(b) Write $\overrightarrow{SC}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{c}$ (1 mark)

$$\overrightarrow{SC} = \underline{c} - \underline{s} \quad \checkmark$$

(c) Use the fact that SC and BA are perpendicular to prove that
$\mathbf{s} \bullet (\mathbf{a} - \mathbf{b}) = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2)$ (4 marks)

$$\therefore \overrightarrow{SC} \cdot \overrightarrow{BA} = 0$$
$$(\underline{c} - \underline{s}) \cdot (\underline{a} - \underline{b}) = 0 \quad \checkmark$$
$$\underline{c} \cdot (\underline{a} - \underline{b}) - \underline{s} \cdot (\underline{a} - \underline{b}) = 0$$
$$s \cdot (\underline{a} - \underline{b}) = \underline{c} \cdot (\underline{a} - \underline{b}) \quad \checkmark$$
$$= \frac{1}{2}(\underline{a} + \underline{b}) \cdot (\underline{a} - \underline{b})$$
$$= \frac{1}{2}(\underline{a} \cdot \underline{a} - \underline{b} \cdot \underline{b}) \checkmark$$
$$= \frac{1}{2}(|\underline{a}|^2 - |\underline{b}|^2) \checkmark$$

Question 16 **(8 marks)**

At 1100 hours at an air show, two planes P and Q are at points whose position vectors are ($50\mathbf{i} - 40\mathbf{j} + 200\mathbf{k}$) km and ($-36\mathbf{i} + 44\mathbf{j} + 186\mathbf{k}$) km respectively relative to an origin O.
P and Q are moving with constant velocities ($-15\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$) km/min and ($10\mathbf{i} - 15\mathbf{j} + 10\mathbf{k}$) km/min respectively.

(a) Find the coordinates of the point at which their trails of smoke meet. (3 marks)

$$\begin{pmatrix} 50 \\ -40 \\ 200 \end{pmatrix} + \lambda \begin{pmatrix} -15 \\ 10 \\ 5 \end{pmatrix} = \begin{pmatrix} -36 \\ 44 \\ 186 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ -15 \\ 10 \end{pmatrix} \checkmark$$

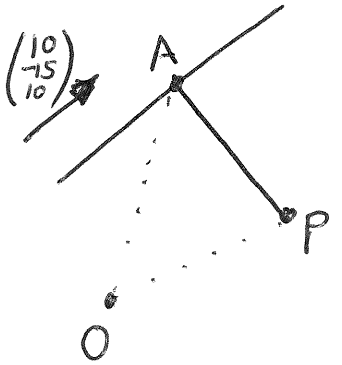
$$\lambda = 3.6$$
$$\mu = 3.2 \checkmark$$

intersect at $\begin{pmatrix} -4 \\ 4 \\ 218 \end{pmatrix}$ ✓

(b) Determine the shortest distance of plane P from the smoke trail of plane Q at 1102 hours. (5 marks)

after 2 mins $r_p(2) = \begin{pmatrix} 50 \\ -40 \\ 200 \end{pmatrix} + 2\begin{pmatrix} -15 \\ 10 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 20 \\ -20 \\ 210 \end{pmatrix}$ ✓



$\overrightarrow{PA} = -\begin{pmatrix} 20 \\ -20 \\ 210 \end{pmatrix} + \begin{pmatrix} -36 \\ 44 \\ 186 \end{pmatrix} + t\begin{pmatrix} 10 \\ -15 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 10t - 56 \\ -15t + 64 \\ 10t - 24 \end{pmatrix}$ ✓

$\therefore \overrightarrow{PA} \cdot \underline{v}_Q = 0$

$\begin{pmatrix} 10t - 56 \\ -15t + 64 \\ 10t - 24 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -15 \\ 10 \end{pmatrix} = 0$ ✓

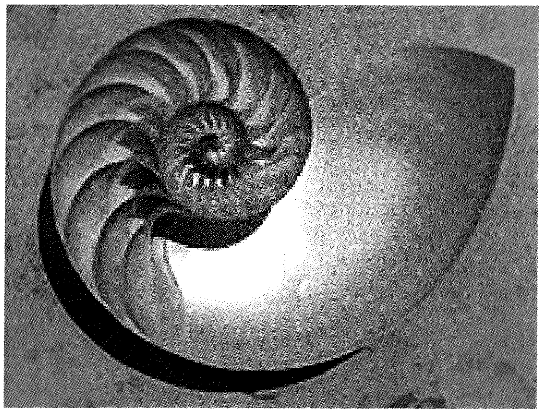
$t = 4.14$ ✓

$\overrightarrow{PA} = \begin{pmatrix} -14.7 \\ 1.9 \\ 17.4 \end{pmatrix}$

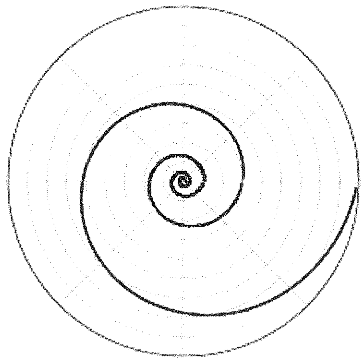
$\therefore |\overrightarrow{PA}| = 22.79$ km ✓

Question 17 **(6 marks)**

The chambers of a nautilus shell are arranged in an approximately logarithmic spiral as shown below.



Nautilus Shell



Logarithmic Sprial

In polar form $(r,\ \theta)$ the logarithmic spiral can be written as $r = e^{k\theta}$.

The length of this spiral in polar form is $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(a) Using $r = e^{k\theta}$ show that

$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \frac{e^{k\theta}\sqrt{k^2+1}}{k} + c$$ (4 marks)

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(e^{k\theta}\right)^2 + \left(k e^{k\theta}\right)^2 \checkmark$$
$$= e^{2k\theta} + k^2 e^{2k\theta}$$
$$= e^{2k\theta}\left(k^2+1\right) \checkmark$$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2k\theta}.(k^2+1)}$$
$$= e^{k\theta}(k^2+1) \checkmark$$

$$\therefore \int\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int e^{k\theta}\sqrt{k^2+1}\, d\theta$$
$$= \frac{e^{k\theta}}{k}.\sqrt{k^2+1} + c \checkmark$$
$$= \frac{e^{k\theta}\sqrt{k^2+1}}{k} + c$$
$$= R.S$$

(b) Given a logarithmic spiral has polar equation $r = e^{0.2\theta}$, find the length of the spiral correct to 2 decimal places for $\theta = 0$ to $\theta = 2\pi$. (2 marks)

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad ✓$$

$$= 12.82 \text{ units} \quad ✓$$